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Angular Distribution of the Zeros of Padé Polynomials*

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INTRODUCTION

Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \qquad (a_0 \neq 0) \tag{1}$$

have a radius of convergence σ_0 $(0 < \sigma_0 \leqslant +\infty)$.

The entries of the Padé table of (1) are ratios of polynomials which may be represented explicitly in terms of the Hankel determinants introduced below.

Let (m, n) be a pair of nonnegative integers; put

$$a_{-j} = 0 \qquad (j = 1, 2, 3,...)$$

and consider the determinants

$$A_m^{(n)} = \begin{vmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n-1} & a_{m+n-2} & a_{m+n-3} & \cdots & a_m \end{vmatrix}, \qquad A_m^{(0)} = 1, \tag{2}$$

and the polynomials

$$D_{mn}(z) = \begin{vmatrix} 1 & z & z^2 & \cdots & z^n \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}, \qquad D_{m0}(z) = 1. \tag{3}$$

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Whenever $A_m^{(n)} \neq 0$, we introduce the *normalized Padé denominator* of the entry (m, n) of the table of (1)

$$Q_{mn}(z) = \frac{D_{mn}(z)}{A_m^{(n)}} = 1 - q_1(m, n) z + \cdots + q_n(m, n) z^n. \tag{4}$$

The corresponding, normalized, Padé numerator is given by

$$P_{mn}(z) = \sum_{j=0}^{m} p_j(m, n) z^j, \tag{5}$$

with

$$p_j(m, n) = \frac{1}{A_m^{(n)}} \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & \cdots & a_{j-n} \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix}. \tag{6}$$

The constant term and leading coefficient of $P_{mn}(z)$

$$p_0(m, n) = a_0, \qquad p_m(m, n) = \frac{A_m^{(n+1)}}{A_m^{(n)}}, \tag{7}$$

play a dominant role in this note.

It is convenient to express the fundamental property of the approximant P_{mn}/Q_{mn} in terms of a contour integral involving an arbitrary polynomial of suitable degree: If $V_{mn}(z)$ is a given polynomial of degree $\leqslant n$, we have

$$\{f(z) Q_{mn}(z) - P_{mn}(z)\} V_{mn}(z) = \frac{z^{m+n+1}}{2\pi i} \int_{\mathscr{C}} \frac{f(\zeta) Q_{mn}(\zeta) V_{mn}(\zeta)}{\zeta^{m+n+1}(\zeta - z)} d\zeta, \tag{8}$$

where we may take for contour $\mathscr{C}$ the circumference $|\zeta| = r$ $(0 < r < \sigma_0)$ described in the positive sense. (A simple deduction of this elementary formula is found in [3, p. 436]).

The Padé polynomials P_{mn}, Q_{mn} are obvious generalizations of the notion of partial sum (or section) of the power series (1). One may expect that the distribution of the zeros of the Padé polynomials is described by theorems analogous to the classical results of Jentzsch, Szegö, Carlson, and Rosenbloom.

I show here that, if $\sigma_0 < +\infty$, that is if the series (1) has a finite radius of convergence, the analogy with the results of Jentzsch and Szegö is complete; I prove

THEOREM 1. *Let $f(z)$, defined by* (1), *be a nonrational function whose radius of meromorphy is τ $(\sigma_0 \leqslant \tau \leqslant +\infty)$. This means that $f(z)$ is mero-*

morphic in the disk $|z| < \tau$ *and also that, if* $\tau < +\infty$, *there is a nonpolar singularity of* $f(z)$ *on the circumference* $|z| = \tau$.

Let $n \geqslant 0$ *be a given integer and let* σ_n $(\leqslant +\infty)$ *be the radius of the largest disk* $|z| < \sigma_n$ *which contains no more than n poles of* $f(z)$ *and none of its nonpolar singulairities.*

Then, if $\sigma_n < +\infty$, *it is possible to find an unbounded sequence* $S(n)$, *of strictly increasing positive integers, which behaves as follows.*

I. $A_m^{(n)} A_m^{(n+1)} \neq 0 \quad (m \in S(n))$.

II. *Consider the normalized Padé numerators* $P_{mn}(z)$. *Given* ϵ $(0 < \epsilon < 1)$ *and* $m \in S(n)$, *there are, in the annulus*

$$(1 - \epsilon)\,\sigma_n \leqslant |z| \leqslant (1 + \epsilon)\,\sigma_n\,,$$

$m(1 - \eta_m)$ *zeros of* $P_{mn}(z)$, *where*

$$0 < \eta_m\,, \qquad \eta_m \to 0 \qquad (m \to \infty).$$

III. *If* $N(m; \varphi_1, \varphi_2)$ *denotes the number of zeros of* $P_{mn}(z)$ *whose arguments lie in the interval*

$$\varphi_1 \leqslant \arg z \leqslant \varphi_2 \qquad (\varphi_1 < \varphi_2 < \varphi_1 + 2\pi),$$

we have

$$\frac{N(m;\, \varphi_1, \varphi_2)}{m} \to \frac{\varphi_2 - \varphi_1}{2\pi} \qquad (m \to \infty,\, m \in S(n)). \tag{9}$$

For $n = 0$, Theorem 1 coincides with Szegö's remarkably precise form of Jentzsch's theorem [13].

If $f(z)$ is entire, the preceding result does not apply because $\sigma_n = +\infty$ for all n. On the other hand, the results of Carlson [1] and Rosenbloom [10] suggest very specific conjectures concerning the behavior of the zeros of the Padé polynomials associated with entire functions of infinite order or of finite, positive order. I have recently established these conjectures and propose to present my results on some other occassion.

It is possible to assert a good deal more about the zeros of the Padé polynomials if one is prepared to assume more about $f(z)$. Results which are interesting and suggestive may be derived from the close study of some special choices of $f(z)$. Among them, the choice

$$f(z) = e^z$$

stands out as particularly important: It was the foundation of Padé's original work and, quite recently, it has led Saff and Varga [12] to results which are a model of elegance and precision.

My proof of Theorem 1 makes essential use of the following elementary property of sequences.

LEMMA I. *Let* $\{\alpha_m\}_{m=1}^{\infty}$ *be a complex sequence such that*

$$\limsup_{m\to\infty} |\alpha_m|^{1/m} = 1. \tag{10}$$

Then, if

$$\liminf_{m\to\infty} |\alpha_m|^{1/m} < 1, \tag{11}$$

there exists an infinite sequence $\mathscr{L}$, *of positive, strictly increasing integers, such that the conditions*

$$m \to \infty, \qquad m \in \mathscr{L}, \tag{12}$$

imply

$$|\alpha_m|^{1/m} \to 1, \qquad |\alpha_m{}^2 - \alpha_{m+1}\alpha_{m-1}|^{1/m} \to 1. \tag{13}$$

The following remarks show that Lemma I, and in particular its very simple proof, may have some independent interest.

PROPOSITION A. *If, in addition to* (10) *we assume*

$$\limsup_{m\to\infty} |\alpha_m{}^2 - \alpha_{m+1}\alpha_{m-1}|^{1/m} < 1, \tag{14}$$

relations (13), *and consequently also* (11), *cannot hold. Hence* (10) *and* (14) *imply*

$$\lim_{m\to\infty} |\alpha_m|^{1/m} = 1. \tag{15}$$

Proposition A may be considered as a restatement of an important lemma of Hadamard [5, pp. 26–28; 2, p. 330, Lemma 2]. Hadamard discovered and used the lemma to establish his fundamental results on polar singularities [5, pp. 24–40; 2, pp. 329–335]. Hadamard's original proof is not very simple and he says about his lemma "c'est le point délicat du raisonnement" [6, p. 80]. More than 30 years after Hadamard's discovery, Ostrowski [7] and Pólya [8] returned to the question and gave new elementary proofs of Hadamard's lemma. I believe that my proof (in Section 3), of the more informative Lemma I, is notably simpler than any of the published proofs of Hadamard's lemma.

I conclude this introduction by stating some notational conventions which are used throughout the paper:

(i) Restrictions such as $m > m_0$, immediately following some relation, mean that the relation in question holds for sufficiently large values of m;

(ii) by $\{\eta_m\}_m$ I denote a sequence such that $\eta_m \to 0$ as $m \to \infty$. The positivity of the η_m's is not assumed;

(iii) it is understood that the sequences $\{\eta_m\}$ and the bounds m_0 are not necessarily the same ones each time they occur.

1. The Sequence of Poles of $f(z)$ and the Polynomials $V_k(z)$

If

$$\sigma_0 = \tau < +\infty,$$

there are no poles of $f(z)$ closer to the origin than the nonpolar singularity or singularities which lie on the circumference $| z | = \tau$.

In all other cases, poles are present and we list explicitly all those which lie in the open disk $| z | < \tau$,

$$b_1, b_2, \ldots, b_k, \ldots. \tag{1.1}$$

The above sequence may be finite or infinite; multiple poles are repeated as often as indicated by their multiplicities and the sequence arranged so that

$$0 < | b_1 | \leqslant | b_2 | \leqslant | b_3 | \leqslant \ldots.$$

If (1.1) has a last element, say b_l, it is convenient to set

$$\infty = b_{l+1} = b_{l+2} = \cdots. \tag{1.2}$$

We define $V_0(z) \equiv 1$, and for all values of $k = 1, 2, 3, \ldots$

$$V_k(z) = \prod_{1 \leqslant j \leqslant k} \left(1 - \frac{z}{b_j}\right) = \sum_{s=0}^{k} v_s(k)\, z^s. \tag{1.3}$$

Whenever necessary, we interpret, with their obvious meaning, the relations (1.3) in the light of the convention (1.2).

It is clear that our definitions always imply

$$\sigma_k = \min\{| b_{k+1} |, \tau\}. \tag{1.4}$$

2. Upper Bounds for the Padé Polynomials

For every value of the integer $k \geqslant 0$, set

$$f_k(z) = f(z)\, V_k(z) = \sum_{j=0}^{\infty} a_j^{(k)} z^j = \sum_{j=0}^{\infty} z^j \left(\sum_{l=0}^{k} a_{j-l} v_l(k)\right). \tag{2.1}$$

In every case, $f_k(z)$ represents a function regular for

$$|z| < \sigma_k \qquad (k = 0, 1, 2, 3,...), \tag{2.2}$$

and hence, by Cauchy's estimate

$$|a_s^{(k)}| (\sigma_k - \epsilon)^s \leqslant \max_{|z|=\sigma_k-\epsilon} \{|f_k(z)|\} = M_k , \tag{2.3}$$

provided

$$\sigma_k < +\infty, \qquad 0 < \epsilon < \sigma_0 , \qquad s, k = 0, 1, 2,....$$

Again, we define $a_s^{(k)} = 0$ for $s < 0$; this enables us to use (2.3) without the restriction $s \geqslant 0$.

My Lemma 2.1 stated and proved below is little more than a rearrangement of the well-known arguments which led Hadamard to his important results on polar singularities. For the convenience of the reader I sketch a brief, self-contained proof.

LEMMA 2.1. *Let $D_{mn}(z)$ be the determinant defined in* (3) *and let*

$$\Delta_{mn}(t) = \max_{|z|\leqslant t} |D_{mn}(z)|. \tag{2.4}$$

Then, for n and t fixed ($1 \leqslant n, 0 \leqslant t$), *we have*

$$\limsup_{m\to\infty}\{\Delta_{mn}(t)\}^{1/m} \leqslant \prod_{j=0}^{n-1} \sigma_j^{-1}, \tag{2.5}$$

and in particular, for $t = 0$,

$$\limsup_{m\to\infty} |A_m^{(n)}|^{1/m} \leqslant \prod_{j=0}^{n-1} \sigma_j^{-1}. \tag{2.6}$$

If $\sigma_{n-1} = +\infty$, the right-hand sides of (2.5) *and* (2.6) *are to be interpreted as* 0.

Proof. We assume, in the following proof, that $\sigma_{n-1} < +\infty$; the slight modifications necessary to cover the case $\sigma_{n-1} = +\infty$ are obvious and are left to the reader. The determinant $D_{mn}(z)$ is of order $n + 1$; we number its rows from 0 to n. To each one of its rows we add a linear combination of the preceding rows. More precisely, to the jth row we add a linear combination of the rows 1, 2, 3,..., $j - 1$, with the respective coefficients

$$v_{j-1}(j-1), v_{j-2}(j-1),..., v_1(j-1).$$

This transforms the jth row of the determinant $D_{mn}(z)$ into

$$a_{m+j}^{(j-1)}, a_{m+j-1}^{(j-1)}, a_{m+j-2}^{(j-1)},..., a_{m+j-n}^{(j-1)} . \tag{2.7}$$

Following this procedure, we modify successively the rows $n, n-1, n-2,\ldots, 2$; the rows 1 and 0 remain unaltered. This leads to a new form $\tilde{D}(z)$ of $D_m^{(n)}(z)$; the rows $1, 2, 3,\ldots, n$ of $\tilde{D}(z)$ are given by (2.7) with $j = 1, 2, 3,\ldots, n$.

Using (2.3), we find

$$| a_{m+l}^{(j-1)} | \leqslant M_{j-1}(\sigma_{j-1} - \epsilon)^{-m-l} \leqslant M_{j-1}(\sigma_{j-1} - \epsilon)^{-m} K^n \qquad (j - n \leqslant l \leqslant j), \tag{2.8}$$

where we may choose

$$K = 1 + (\sigma_0 - \epsilon)^{-1} + \sigma_{n-1}.$$

It is no restriction to assume $t \geqslant 1$. Hence, for a suitable value of z_0 ($| z_0 | = t$), we have, in view of (2.8)

$$\begin{aligned} | D_{mn}(z_0)| &= \Delta_{mn}(t) = | \tilde{D}(z_0)| \\ &\leqslant (n + 1)!\, t^n K^{n^2} M_0 M_1 M_2 \cdots M_{n-1} \prod_{j=0}^{n-1} (\sigma_j - \epsilon)^{-m}. \end{aligned} \tag{2.9}$$

Hence with t, n, and ϵ fixed, (2.9) yields

$$\limsup_{m\to\infty} \{\Delta_{mn}(t)\}^{1/m} \leqslant \prod_{j=0}^{n-1} (\sigma_j - \epsilon)^{-1}. \tag{2.10}$$

Since the left-hand side of (2.10) is independent of ϵ, we deduce (2.5) from (2.10) by letting $\epsilon \to 0+$. This completes the proof of Lemma 2.1.

LEMMA 2.2. *Let the value of the integer $n \geqslant 0$ be fixed, let $\sigma_n < +\infty$ and let the values of m be restricted to some infinite sequence $\mathscr{L}$ of strictly increasing positive integers such that*

$$| A_m^{(n)} |^{1/m} \to \prod_{j=0}^{n-1} \sigma_j^{-1} \qquad (m \to \infty,\ m \in \mathscr{L},\ n \geqslant 1). \tag{2.11}$$

[For $n = 0$ our definitions imply $A_m^{(0)} = 1$ and (2.11) *takes the trivial form* $| A_m^{(0)} |^{1/m} \to 1$ as $m \to \infty$.]

Put

$$\Omega_m = \max_{|z|=\sigma_n} | P_{mn}(z)|. \tag{2.12}$$

Then

$$\Omega_m^{1/m} \to 1 \qquad (m \to \infty,\ m \in \mathscr{L}). \tag{2.13}$$

Proof. Since

$$\sigma_0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n < +\infty,$$

it is clear that

$$A_m^{(n)} \neq 0 \qquad (m > m_0,\, m \in \mathscr{L}).$$

Hence, by (4), (2.4), (2.5), and (2.11)

$$\max_{|z|=\sigma_n} |Q_{mn}(z)| = \Lambda_{mn} \leqslant \Delta_{mn}(\sigma_n)/|A_m^{(n)}|, \tag{2.14}$$

$$\Lambda_{mn} \leqslant (1 + \eta_m)^m \qquad (\eta_m \to 0,\, m \to \infty,\, m \in \mathscr{L}). \tag{2.15}$$

(The above relation is trivial if $n = 0$.)

In the remainder of the proof n is fixed and we simplify our notation by writing

$$P_m,\, Q_m,\, V,\, \Lambda_m,\, \sigma,\, p_j(m),$$

instead of

$$P_{mn},\, Q_{mn},\, V_n,\, \Lambda_{mn},\, \sigma_n,\, p_j(m, n).$$

Select ϵ $(0 < 3\epsilon < \sigma)$ such that $V(z)$ has no zeros in the annulus

$$\sigma - 3\epsilon < |z| < \sigma;$$

ϵ is otherwise arbitrary. Clearly

$$\min_{|z|=\sigma-2\epsilon} |V(z)| \geqslant (\epsilon/\sigma)^n. \tag{2.16}$$

We now introduce $\tilde{M}$ (necessarily finite) by the relation

$$\max_{|\zeta|\leqslant\sigma-\epsilon} |f(\zeta)\, V(\zeta)| = \tilde{M}. \tag{2.17}$$

Consider (8) and notice that, since $f(z)\, V(z)$ is regular for $|z| < \sigma$, we may use for contour $\mathscr{C}$ the circumference

$$|\zeta| = \sigma - \epsilon.$$

By elementary estimates, (8), (2.16), and (2.17) now yield

$$\min_{|z|=\sigma-2\epsilon} |V(z)| \max_{|z|=\sigma-2\epsilon} |P_m(z)| \leqslant (1 + \sigma/\epsilon)\, \tilde{M}\Lambda_m,$$

$$\max_{|z|=\sigma-2\epsilon} |P_m(z)| \leqslant 2\epsilon^{-n-1}\sigma^{n+1}\tilde{M}\Lambda_m. \tag{2.18}$$

This inequality and Cauchy's estimate imply

$$|p_j(m)|\, \sigma^j \leqslant (\sigma/(\sigma - 2\epsilon))^j\, 2\epsilon^{-n-1}\sigma^{n+1}\tilde{M}\Lambda_m \qquad (j = 0, 1, 2,\ldots, m),$$

and hence, by (2.12) and (2.15),

$$\Omega_m \leqslant 2(m+1)\,\epsilon^{-n-1}\sigma^{n+1}\tilde{M}(1+\eta_m)^m\,(\sigma/(\sigma-2\epsilon))^m \qquad (m \in \mathscr{L}),$$

$$\limsup_{\substack{m\to\infty \\ m\in\mathscr{L}}} \Omega_m^{1/m} \leqslant \sigma/(\sigma-2\epsilon).$$

The left-hand side of this relation is independent of ϵ and consequently, letting $\epsilon \to 0+$, we may replace its right-hand side by 1. Since

$$\Omega_m \geqslant |\,P_m(0)| = |\,a_0\,| \neq 0,$$

we immediately obtain (2.13) and thus complete the proof of Lemma 2.2.

3. Lemma I and its Consequences

Proof of Lemma I. Let $L > 1$ and $\epsilon > 0$ be given. Assume that ϵ is small enough to imply

$$\liminf_{m\to\infty} |\,\alpha_m\,|^{1/m} < 1-\epsilon; \tag{3.1}$$

there are no further restrictions on ϵ.

Since $(1/m)^{1/m} \to 1$ as $m \to \infty$, we also have, by (10) and (3.1)

$$\limsup_{m\to\infty} \left|\frac{\alpha_m}{m}\right|^{1/m} = 1, \qquad \liminf_{m\to\infty} \left|\frac{\alpha_m}{m}\right|^{1/m} < 1-\epsilon. \tag{3.2}$$

From (3.2) we deduce the existence of three integers j, k, l such that

$$L < j < k < l$$

and

$$\left|\frac{\alpha_j}{j}\right|^{1/j} < 1-\epsilon, \qquad \left|\frac{\alpha_k}{k}\right|^{1/k} > 1-\epsilon, \qquad \left|\frac{\alpha_l}{l}\right|^{1/l} < 1-\epsilon.$$

Then there must exist some integer m such that

$$L < j < m < l, \qquad \left|\frac{\alpha_m}{m}\right|^{1/m} = \max_{j\leqslant s\leqslant l} \left|\frac{\alpha_s}{s}\right|^{1/s}.$$

Hence the four following inequalities hold simultaneously:

$$\left|\frac{\alpha_m}{m}\right|^{1/m} > 1-\epsilon, \qquad m > L, \tag{3.3}$$

$$\left|\frac{\alpha_m}{m}\right|^{1/m} \geqslant \left|\frac{\alpha_{m-1}}{m-1}\right|^{1/(m-1)}, \qquad \left|\frac{\alpha_m}{m}\right|^{1/m} \geqslant \left|\frac{\alpha_{m+1}}{m+1}\right|^{1/(m+1)}. \tag{3.4}$$

By (3.4) and (3.3)

$$|\alpha_m^2 - \alpha_{m+1}\alpha_{m-1}| \geq |\alpha_m|^2 \left\{1 - \left(1 - \frac{1}{m^2}\right)\right\} > (1-\epsilon)^{2m}, \quad m > L. \quad (3.5)$$

We now use the preceding inequalities to define, successively, the members $m(j)$ $(j = 1, 2, 3,...)$ of the sequence $\mathscr{L}$ of Lemma I. Assume that $m(j)$ $(j = 1, 2,..., N)$ have been determined. In view of the arbitrary character of ϵ and L we may use (3.5) with

$$\epsilon = \frac{1}{N+1}, \qquad L = m(N).$$

Hence (3.5) enables us to select $m = m(N+1)$ such that

$$|\alpha_m^2 - \alpha_{m+1}\alpha_{m-1}|^{1/m} > \left(1 - \frac{1}{N+1}\right)^2,$$

$$|\alpha_m|^{1/m} > 1 - \frac{1}{N+1}, \qquad m(N+1) > m(N).$$

This concludes our proof of Lemma I since it is now obvious that the sequence

$$\mathscr{L} = \{m(N)\}_{N=1}^{\infty}$$

has the properties expressed by (12) and (13).

4. Selection of the Sequence $S(n)$

We start from the relations

$$\limsup_{m\to\infty} |A_m^{(n)}|^{1/m} = \prod_{j=0}^{n-1} \sigma_j^{-1} = h_n \qquad (n = 1, 2, 3,...), \qquad (4.1)$$

which follow, by induction, from the reasoning in [2, p. 334]. The formulation (4.1), which is the most convenient one for my purpose, is clearly equivalent to the fundamental results of Hadamard on polar singularities [2, pp. 329–335].

LEMMA 4.1. *Let the assumptions and notations of Theorem* 1 *be satisfied. Let the integer* $n \geq 1$ *be given and let* $\sigma_n < +\infty$.

It is then possible to determine an infinite sequence $S(n)$, *of strictly increasing positive integers such that, as*

$$m \to \infty, \qquad m \in S(n), \qquad (4.2)$$

the two relations

$$| A_m^{(n)} |^{1/m} \to \prod_{j=0}^{n-1} \sigma_j^{-1}, \qquad | A_m^{(n+1)} |^{1/m} \to \prod_{j=0}^{n} \sigma_j^{-1}, \tag{4.3}$$

simultaneously hold.

Proof. From

$$0 < \sigma_0 \leqslant \sigma_j \leqslant \sigma_n < +\infty \qquad (0 \leqslant j \leqslant n), \tag{4.4}$$

and (4.1) we see that

$$0 < h_j < +\infty \qquad (1 \leqslant j \leqslant n+1).$$

Write

$$h_n = h,$$

and introduce the sequences $\{\alpha_m\}$, $\{\gamma_m\}$, $\{\delta_m\}$ defined by

$$\alpha_m = h^{-m} A_m^{(n)}, \qquad \gamma_m = h^{-m} A_m^{(n+1)}, \qquad \delta_m = h^{-m} A_m^{(n-1)}. \tag{4.5}$$

From the well-known identity

$$\{A_m^{(n)}\}^2 - A_{m+1}^{(n)} A_{m-1}^{(n)} = A_m^{(n+1)} A_m^{(n-1)} \qquad (n \geqslant 1, m \geqslant 0)$$

[9, p. 102, Example 19] (the notation of Pólya and Szegö differ from the ones adopted here), we deduce

$$G_m = | \alpha_m{}^2 - \alpha_{m+1}\alpha_{m-1} | = | \gamma_m || \delta_m | , \tag{4.6}$$

and from (4.1), (4.4), and (4.5)

$$\limsup_{m\to\infty} | \alpha_m |^{1/m} = 1, \tag{4.7}$$

$$\limsup_{m\to\infty} | \gamma_m |^{1/m} = \frac{1}{\sigma_n}, \tag{4.8}$$

$$\limsup_{m\to\infty} | \delta_m |^{1/m} = \sigma_{n-1} . \tag{4.9}$$

From (4.6) and (4.7) we also conclude

$$\limsup_{m\to\infty} G_m^{1/m} = \limsup_{m\to\infty} | \gamma_m \delta_m |^{1/m} = \chi \leqslant 1. \tag{4.10}$$

Now there are two possibilities; either

$$\chi < 1 \tag{4.11}$$

or

$$\chi = 1. \tag{4.12}$$

Assume first that (4.11) *is satisfied.* Then, (4.7), (4.10), (4.11), and Hadamard's lemma [stated as Proposition A of the Introduction] show that (15) holds. We next use (4.8), and see that there exists an infinite sequence $S(n)$ such that

$$|\gamma_m|^{1/m} \to 1/\sigma_n \qquad (m \to \infty, m \in S(n)). \tag{4.13}$$

Hence, as a trivial consequence of (15),

$$|\alpha_m|^{1/m} \to 1 \qquad (m \to \infty, m \in S(n)). \tag{4.14}$$

Using (4.5) we express (4.13) and (4.14) in terms of $A_m^{(n+1)}$, $A_m^{(n)}$ and immediately obtain (4.3).

If (4.11) *does not hold,* (4.12) *must be satisfied.* Now either (15) holds or we may use Lemma I (stated in the Introduction) and take $S(n) = \mathscr{L}$. With this new meaning of $S(n)$, (4.14) still holds and, by (13) and (4.6),

$$|\gamma_m|^{1/m} |\delta_m|^{1/m} \to 1 \qquad (m \to \infty, m \in S(n)),$$

$$1 \leqslant (\liminf_{\substack{m\to\infty \\ m\in S(n)}} |\gamma_m|^{1/m})(\limsup_{m\to\infty} |\delta_m|^{1/m}). \tag{4.15}$$

Hence, if the lim inf in the above relation is $<1/\sigma_n$, we must, in view of (4.9) have

$$\sigma_{n-1} > \sigma_n .$$

This contradicts (4.4) and, consequently, (4.13) as well as (4.14) must hold with the new meaning of $S(n)$. As in the preceding argument we thus establish (4.3). The proof of Lemma 4.1 is now complete.

5. Proof of Theorem 1

Consider first the case $n \geqslant 1$; n is a given integer, $S(n)$ is the infinite sequence whose existence is established in Lemma 4.1, and

$$\begin{aligned} P_{mn}(z) = P_m(z) &= a_0 + p_1(m, n)\, z + \cdots + p_m(m, n)\, z^m \\ &= a_0 \prod_{j=1}^{m} \left(1 - \frac{z}{z_j(m, n)}\right) \qquad (m \in S(n)) \end{aligned} \tag{5.1}$$

defines the associated sequence of Padé numerators.

The values of the coefficients $p_j(m, n)$ are explicitly given by (6). For our purpose it is essential to note the value of the leading coefficient $p_m(m, n)$ given in (7).

Since n remains fixed throughout the proof we omit it from the notation unless the omission is likely to create confusion. We write below

$$S,\ P_m,\ p_m(m),\ z_j(m),\ \sigma$$

instead of

$$S(n),\ P_{mn},\ p_m(m, n),\ z_j(m, n),\ \sigma_n .$$

We perform the change of variable

$$t = z/\sigma \tag{5.2}$$

and consider beside $P_m(z)$ the auxiliary polynomial $T_m(t)$ of the variable t

$$T_m(t) = P_m(\sigma t) = a_0 + p_1(m)\,\sigma t + \cdots + p_m(m)\,\sigma^m t^m = \sum_{j=0}^{m} t_j(m)\,t^j. \tag{5.3}$$

By Lemma 4.1 we have as

$$m \to \infty, \qquad m \in S, \tag{5.4}$$

the limit relations

$$|A_m^{(n)}|^{1/m} \to \prod_{j=0}^{n-1} \sigma_j^{-1}, \tag{5.5}$$

$$\left|\frac{A_m^{(n+1)}}{A_m^{(n)}}\right|^{1/m} \to \frac{1}{\sigma}. \tag{5.6}$$

Now by (5.3)

$$\max_{|t|=1} |T_m(t)| = \max_{|z|=\sigma} |P_m(z)| = \Omega_m , \tag{5.7}$$

and by (5.5) and Lemma 2.2,

$$\log \Omega_m = o(m) \qquad (m \to \infty,\ m \in S). \tag{5.8}$$

From this point on, the proof of assertion III of Theorem 1 may be obtained by using a simple and elegant result of Rosenbloom's [10, Theorem XIII, p. 25]. To avoid a proof based on some work which may be difficult to consult, we use, instead, a classical theorem of Erdös and Turan [4]. The latter result, published several years after Rosenbloom's thesis, is unnecessarily precise for the purpose of the present note. Rosenbloom restated and generalized in [11] many results of his dissertation. Since, in his later version, Rosenbloom only sketches most of his proofs, it may be that [4] still provides the easiest access to complete arguments.

In order to apply the result of Erdös and Turan we consider the expression

$$X_m = \frac{\sum_{j=0}^{m} | t_j(m)|}{| t_0(m)\, t_m(m)|^{1/2}} .$$

By (7), (5.3), and (5.6)

$$| t_0(m)\, t_m(m)| = | a_0 | \sigma^m | p_m(m)| = | a_0 |(1 + \eta_m)^m, \tag{5.9}$$

with $\eta_m \to 0$ under the conditions (5.4).

From (5.7), Cauchy's estimate, and (5.8) we deduce

$$\log \left(\sum_{j=0}^{m} | t_j(m)| \right) \leqslant \log(m + 1) + \log \Omega_m = o(m),$$

and hence, in view of (5.9)

$$\log X_m = \eta_m m.$$

(In the preceding estimates we have used our notational convention about η_m.)

The theorem of Erdös and Turan [4, Theorem 1, p. 106] now asserts

$$\left| N(m; \varphi_1, \varphi_2) - \frac{(\varphi_2 - \varphi_1)}{2\pi} m \right| < 16 m^{1/2} (\log X_m)^{1/2}.$$

This clearly implies (9) and assertion III of Theorem 1 is proved.

To prove assertion II of the Theorem denote by ν_m the number of zeros of $T_m(t)$ in the disk

$$| t | \leqslant 1 - \epsilon \qquad (0 < \epsilon < 1).$$

Then, by Jensen's formula

$$\log | a_0 | + \nu_m \log \left(\frac{1}{1 - \epsilon} \right)$$
$$\leqslant \log | a_0 | + \sum_{(|z_j(m)|/\sigma) \leqslant 1-\epsilon} \log \left\{ \frac{1}{| z_j(m)|/\sigma} \right\} \leqslant \log \Omega_m$$

and hence by (5.8)

$$\nu_m = o(m) \qquad (m \to \infty, m \in S). \tag{5.10}$$

Similarly, considering the polynomial

$$W_m(w) = w^m T_m(1/w)$$

instead of $T_m(t)$, we see that the number $\tilde{\nu}_m$ of zeros of $P_m(z)$ in the region

$$| z | \geqslant \sigma(1 + \epsilon)$$

satisfies the condition

$$\tilde{\nu}_m = o(m) \qquad (m \to \infty, m \in S). \tag{5.11}$$

Assertion II of Theorem 1 follows from (5.10) and (5.11). This completes the proof of Theorem 1 in the case $n \geqslant 1$.

The case $n = 0$ is of interest since it corresponds to the original theorem of Jentzsch and Szegö. In order to cover it by the above method, it suffices to observe that, since $A_m^{(0)} = 1$, we may select $S(0)$ to be any infinite sequence such that

$$| A_m^{(1)} |^{1/m} = | a_m |^{1/m} \to 1, \qquad (m \to \infty, m \in S(0)).$$

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